The sustitution properties of a production frontier demand curve

Essay in the honor of Arnold C. Harberger in the ocassion of his 71^{st} birthday.

> "It is easy to show that the Hicks-Slutsky substitution properties apply to demand functions defined by movements constrained to a locus of the form: $\Sigma C_i X_i = Y$, a constant, so long as one is concerned with small changes in the neighborhood of the undistorted equilibrium. However, this cannot be shown to be generally true for large changes."

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To explore the rationale of this assertion, which seems to be intuitively obvious, we shall consider a world of two goods X_1, X_2 and a consumer with an utility function of the form:

1) $U(X_1, X_2) = \alpha \log X_1 + ß \log X_2 ; \quad (\alpha + ß) = 1.$

Demand functions of the type under consideration can be obtained by solving the following system of equations:

2) $Y^0 = X_1 P_1 + X_2 P_2$

3) $Y^0 = X_1 + X_2$

4) $\delta P_1 = \alpha / X_1$

5) $\delta P_2 = ß / X_2$

This is a system of four equations with five unknowns: X_1, X_2, P_1, P_2 and δ. Prices are endogenous to the system. P_1 and P_2 are real prices, that is current prices deflated by a Paasche price index, so that whenever different situations are compared private expenditure (Y^0) remains constant (equation (2)). Equation (3) depicts the production opportunities assumed to be linnear. Equation (4) and (5) are the consumer's marginal conditions of equilibrium for the chosen utility function (1).

By solving for X_1 and X_2 we get their demand functions and a price frontier:

6) $$X_1 = \frac{P_2}{P_2 + \Theta P_1} \cdot Y^0$$

$$7)\quad X_2 = \frac{\Theta P_1}{P_2 + \Theta P_1} \cdot Y^0$$

$$8)\quad 0 = (1 + \Theta) P_1 P_2 - \Theta P_1 - P_2$$

Where $\Theta = \beta/\alpha$.

The hicksian substitution properties are usually depicted in terms of the partial derivatives:

a) $S_{ii} < 0$; b) $S_{ij} = S_{ji}$; c) $\Sigma S_{ij} P_j = 0$

When working with production frontier demand curves S_{ij} stands for dx_i/dp_j

Let us consider the total differential dX_1:

$$9)\quad dX_1 = \frac{\delta X_1}{\delta P_1} \cdot dP_1 + \frac{\delta X_1}{\delta P_2} dP_2$$

this expression can be written in elasticity form:

$$10)\quad \eta_{11} = \bar{\eta}_{11} + \bar{\eta}_{12} \quad \eta_{P1\ P2}$$

Where η_{11} is the elasticity of the demand for X_1 with respect to a change in its own price cum the corresponding change in the price of the other good; $\bar{\eta}_{11}$ and $\bar{\eta}_{12}$ are partial elasticities and $\eta_{P2\ P1}$ is the elasticity of P_2 with respect to P_1 along the Price Frontier (8). For the demand equation (6) these values are:

$$\bar{\eta}_{11} = - \frac{\Theta P_1}{P_2 + \Theta P_1} \quad ; \quad \bar{\eta}_{12} = - \bar{\eta}_{11} \quad ; \quad \eta_{P2\ P1} = - \frac{P_2}{\Theta P_1}$$

Putting these values in equation (10) we obtain a value of $\eta_{11} = - 1$. This value implies that the hiksian condition $S_{11} < 0$ is satisfied.

To examine the property $S_{ij} = S_{ji}$ we compute the elasticities η_{12} and η_{21} using the following relationships:

$$11)\quad \eta_{12} = \bar{\eta}_{12} + \bar{\eta}_{11} \quad \eta_{P1\ P2}$$

12) $\eta_{21} = \bar{\eta}_{21} + \bar{\eta}_{22}\ \eta_{P2\ P1}$

Since $\eta_{P1\ P2} = \dfrac{1}{\eta_{P2\ P1}}$ equations (11) and (12) can be easily estimated: $\eta_{12} = \Theta\ P_1/P_2$ and $\eta_{21} = P_2/\Theta P_1$. To compute the values S_{12} and S_{21} we use the following relationship:

$$13)\quad \eta_{12} = \frac{d\ X_1}{d\ P_2}\ \frac{P_2}{X_1} = \frac{S_{12}\ P_2}{X_1} = \frac{\Theta\ P_1}{P_2}$$

$$14)\quad S_{12} = \frac{\Theta\ P_1\ X_1}{P^2_2}$$

Using the same procedure we get the value $S_{21} = \dfrac{X_2\ P_2}{\Theta\ P^2_1}$ whence the condition $S_{12} = S_{21}$ is not satisfied by this Production Frontier Demand Curve.

Let us consider now the property $\Sigma\ S_{ij}\ P_j = 0$. This property can be expressed in elasticity form:

$$15)\quad \Sigma\ S_{ij}\ P_j = X_1\ \frac{\Sigma\ S_{ij}\ P_j}{X_1} = X_1\ (\eta_{11} + \eta_{12})$$

Replacing $\eta_{11} = -1$ and $\eta_{12} = \Theta P_1/P_2$ in this expression we get:

$$16)\quad \Sigma\ S_{ij}\ P_j = X_1\ (-1 + \frac{\Theta\ P_1}{P_2})$$

an expression that in general will not be zero. Hence the hicksian condition $\Sigma\ S_{ij}\ P_j = 0$ is not satisfied by our Production Frontier Demand Curve.

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