The Problem of the Substitution Properties of Production Frontier Demand Curves

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It is widely known that the substitution term of demand curves derived under the hypothesis postulated by Hicks in his contribution of 1938 do satisfy a set of conditions. The purpose of this paper is to explore the applicability of these conditions to demand curves derived under a set of conditions that differs from the ones chosen by Hicks in his seminal paper. To illustrate the reader upon the nature of this problem we will develop the case of a consumer in a world of two goods by comparing two different demand curves: one derived under the set of conditions in which the hicksian properties apply and the other a production frontier demand curve which is a demand curve derived under an alternative set of assumptions.

The standard case is illustrated by considering the derivation of a demand curve of a consumer in a world of two goods X_1, X_2 with an utility function of the form: $U(X_1, X_2) = \alpha \log X_1 + \beta \log X_2$, $(\alpha + \beta = 1)$, with a (nominal) income I and facing prices P_1 and P_2 that are exogenous to the system.

The demand curves will be a function of these exogenous prices and can be obtained by solving the following system of equations:

1) $I = X_1 P_1 + X_2 P_2$

2) $\delta P_1 = \alpha / X_1$

3) $\delta P_2 = \beta / X_2$

Solving for X_1 and X_2 we obtain two demand curves:

4) $X_1 = \alpha I P_1^{1}$

5) $X_2 = \beta I P_2^{1}$

These demands show the uncompensated response of a consumer to changes in the prices P_1 and P_2 while his nominal income I remains constant. By the same token the implied elasticities of the demand for X_1 and X_2, which in both cases are equal to -1, do reflect this uncompensated response. To obtain the elasticities that capture the pure substitution effect of a change in the price of one of the goods we must resort to Slutsky's fundamental equation expressed in elasticity form:

6) $\eta^{S}_{ij} = \eta_{ij} + f_j \cdot \varepsilon_i$

With this formula we can compute an elasticity of demand that captures the pure substitution effect. Let us consider the elasticity of demand for X_1. In this case $i = j = 1$. $f_1 = \alpha$ and $\varepsilon_1 = 1$.

7) $\eta^{S}_{11} = \eta_{11} + \alpha$

This means that $\eta^{S}_{11} = -1 + \alpha$ and by the same argument $\eta^{S}_{22} = -1 + \beta$.

The hicksian substitution properties are usually depicted in terms of the partial derivatives:

a) $S_{ii} < 0$; b) $S_{ij} = S_{ji}$; c) $\sum_j S_{ij} P_j = 0$

These set of conditions can also be expressed in elasticity form. Condition a) can be expressed alternatively as $\eta^{S}_{ii} < 0$ which is satisfied in our case. Condition b) can be expressed as:

8) $X_1 \cdot P_1 \, \eta^{S}_{12} = X_2 \, P_2 \, \eta^{S}_{21}$

The values of η^{S}_{12} and η^{S}_{21} can be computed by using equation 6):

9) $\eta^{S}_{12} = \eta_{12} + \beta = 0 + \beta$

10) $\eta^{S}_{21} = \eta_{21} + \alpha = 0 + \alpha$

Replacing the values of η^{S}_{12} and η^{S}_{21} in 8) we get:

11) $X_1 \cdot P_1 \, \beta = X_2 \cdot P_2 \, \alpha$

Since $X_1 \cdot P_1 = \alpha I$ and $X_2 \cdot P_2 = \beta I$ condition b) is also satisfied.

Condition c) can also be expressed in elasticity form. Let us consider the demand for X_1:

12) $\sum_j S_{ij} P_j = X_i \left(\Sigma \eta^{S}_{ij} \right) = X_1 \, (-1 + \alpha + 0 + \beta) = 0$

and condition c) is satisfied.

It should not come as a surprise to find out the hicksian firt order conditions being satisfied in the exercise of above. These properties have been shown to hold under more general conditions concerning the nature of the utility function as well as the number of commodities involved. The moral of this exercise is that it provides an analytic framework to explore the properties of demand curves generated under a set of conditions that differ from the ones chosen by Hicks in his theoretical development. The Production Frontier Demand Curve is a case in point.

To illustrate the reader on this point we will consider the same utility function as before and the theoretical model that leads to demand functions of this type.

13) $Y^{\circ} = X_1 P_1 + X_2 P_2$

14) $Y^{\circ} = X_1 + X_2$

15) $\delta P_1 = \alpha / X_1$

16) $\delta P_2 = \beta / X_2$

This model differs from the one chosen by Hicks in many aspects. It is a general equilibrium model that focusses on a consumer that is a replica of the consumers of a community in both tastes and preferences (equations 15) and 16)) as well as in his production opportunities (equation 14)). The prices are endogenous to the system. P_1 and P_2 are real prices, that is current prices deflated in such a way that whenever different situations are being compared private expenditure (Y°) remains constant (equation 13)). The fact that in equilibrium the consumer is bound to satisfy the production opportunities depicted by equation (14) has been the origin of the name given to the demand curves that this model generates: Production Frontier Demand Curves.

There are some difficulties in working out analytical solutions to this system of equations due to the fact that the variables enter non linearly in some of the equations.

Solving for X_1 and X_2 we get their demand functions and a price frontier:

17) $$X_1 = \frac{P_2}{P_2 + \Phi P_1} \cdot Y^{\circ}$$

18) $$X_2 = \frac{\Phi P_1}{P_2 + \Phi P_1} \cdot Y^{\circ}$$

19) $$0 = (1 + \Phi) P_1 P_2 - \Phi P_1 - P_2$$

where $\Phi = \beta / \alpha$.

This system of equations presents some features worth mentioning. An important one is the fact that real private expenditure (Y°) remains constant. Another feature is that when a change in the price of a commodity is being considered the corresponding change in the price of the other commodity must be allowed for. Under the conditions of the problem these prices are related according function 19) usually known as the Price Frontier. If in addition to this we remember that the equilibrium is constrained to the production opportunities locus 14) we can easily conclude that we are dealing with situations in which "real income", in the sense of the range of alternative production

ΦP_1^2

Using the same procedure we obtain the value $S_{21} = X_2 \; P_2 / \Phi P_1^2$, ~~and~~ whence we can conclude that the condition $S_{12} = S_{21}$ is not satisfied by this Production Frontier Demand Curve.

Let us consider now property c) $\Sigma S_{1j} P_j = 0$. This property can be expressed in elasticity form:

$$\Sigma S_{1j} P_j = X_1 \Sigma \frac{S_{1j} P_j}{X_1} = X_1 \cdot (\eta_{11} + \eta_{12})$$

We already know that $\eta_{11} = -1$ and $\eta_{12} = \Phi \; P_1/P_2$. Replacing these values in the expression of above we get:

$$\Sigma \; S_{1j} P_j = X_1 \left(-1 + \frac{\Phi \; P_1}{P_2} \right)$$

an expression that in general will not be zero. Hence condition c) $\Sigma \; S_{1j} P_j = 0$ is not fullfilled by this Production Frontier Demand Curve.

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oportunities, remains constant. This leaves us in a world of pure substitution effects where the difference between compensated or uncompensated demand curves is meaningless.

With these arguments in mind, let us consider the total differential dX_1:

$$20)\quad dX_1 = \frac{\partial X_1}{\partial P_1} dP_1 + \frac{\partial X_1}{\partial P_2} dP_2$$

this expression can be expressed in elasticity form:

$$21)\quad \eta_{11} = \bar{\eta}_{11} + \bar{\eta}_{12} \cdot \eta_{P_2P_1} \rightarrow \eta_{P_2P_1}$$

Where η_{11} is the elasticity of the demand for X_1 with respect to a change in its own price (cum the corresponding change in the price of the other good); $\bar{\eta}_{11}$ and $\bar{\eta}_{12}$ are partial elasticities and $\eta_{P_2P_1}$ is the elasticity of P_2 with respect to P_1 along the Price Frontier 19). For the demand equation 17) these values are: $\eta_{P_2P_1}$

$$\bar{\eta}_{11} = -\frac{\Phi P_1}{P_2 + \Phi P_1};\quad \bar{\eta}_{12} = -\bar{\eta}_{11};\quad \eta_{P_2P_1} = -\frac{P_2}{\Phi P_1}$$

Putting these values in equation 21) we obtain a value $\eta_{11} = -1$. This is the elasticity value that should be compared to the one that captures the pure substitution effect in the hiksian world: $\eta^s_{11} = -1 + \alpha$, a value that is obviously different. This raises the issue of the applicability of the hiksian substitution properties to Production Frontier Demand Curves.

It is evident that condition $S_{11} < 0$ is satified by the production Frontier Demand Curve even when its elasticity in absolute value differs from the one obtained under the hypothesis postulated by Hicks.

To examine property b) $S_{12} = S_{21}$ we compute the elasticities η_{12} and η_{21} using the following relationships:

$$22)\quad \eta_{12} = \bar{\eta}_{12} + \bar{\eta}_{11} \cdot \eta_{P_1P_2}$$

$$23)\quad \eta_{21} = \bar{\eta}_{21} + \bar{\eta}_{22} \cdot \eta_{P_2P_1}$$

$\frac{1}{\eta_{P_2P_1}}$

Since $\eta_{P_1P_2} = \frac{1}{\eta_{P_2P_1}}$ equations 22) and 23) can be easily estimated. Thus,[1] the values $\eta_{12} = \Phi P_1/P_2$ and $\eta_{21} = P_2/\Phi P_1$ are obtained. To compute the values S_{12} and S_{21} we use the following relationship:

$$\eta_{12} = \frac{dX_1}{dP_2} \cdot \frac{P_2}{X_1} = \frac{S_{12} P_2}{X_1} = \frac{\Phi P_1}{P_2}$$

$$S_{12} = \frac{\Phi P_1 X_1}{P_2^2}$$

1) A positive cross-elasticity of demand in a world of two goods is mandato and the result satisfy this condition. ~~Let us consider now these~~

$P_1 = P_2 = 1$

$$\frac{dX_1}{dP_2} = S_{12} = \frac{1/2\,Y^o}{1} = \frac{1}{2}Y^o = \frac{1}{2}Y^o$$

$$\frac{dX_2}{dP_1}\quad S_{21} = \frac{X_2 P_2}{P_1^2} = \frac{0{,}5834 \cdot Y^o \times 0.857}{1.2^2} = \frac{1/2\,Y^o}{1.44}$$

$$\frac{dX_1}{dP_2}\quad S_{12} = \frac{P_1 X_1}{P_2^2} = \frac{1/2\,Y^o}{0{,}7344}$$

ΦP_1^2

whence

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